GENERAL DECAY RATES FOR TOTAL ENERGY OF A VISCOELASTIC WAVE EQUATION WITH STRONG DAMPING AND VARIABLE SOURCES

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ABSTRACT

This paper focuses on the general decay rates for total energy associated with the solution of the nonlinear viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-\kappa)\Delta u(\kappa)d\kappa - \Delta u_t = |u|^{p(x)-2} u$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, under the assumption $g'(t) \le -\xi(t)G(g(t))$ on the relaxation function together with sub-critical initial energy. These refine and extend our recent results about viscoelastic wave equations.

Keywords: Nonlinear wave equation, Viscoelasticity, General decay, Strong damping, Variable exponent sources.

1. INTRODUCTION

The goal herein is to extend the decay results from the study in [1] for the nonlinear viscoelastic wave equation with variable exponents and subject to strong damping

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\kappa)\Delta u(\kappa)d\kappa + h(u_t) = f(u), & \text{in} \quad \Omega \times (0,T), \\ u = 0, & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $f(u) = |u|^{p(x)-2} u$, $T > 0$, $\Omega \subset \mathbb{R}^n (n \ge 2)$ is a bounded domain and the boundary $\partial\Omega$ is smooth, the damping term $h(u_t) = -\Delta u_t$, the initial data u_0 and u_1 are known, g is positive, non-increasing and continuously differentiable. The exponent $p(x)$ is continuous on Ω satisfying the following conditions:

$$2 < p^- = \inf_{x\in\bar{\Omega}} p(x) \le p(x) \le p^+ = \sup_{x\in\bar{\Omega}} p(x) \le \frac{2(n-1)}{n-2}, \quad n \ge 3, \tag{1.2}$$

$$\forall x, y \in \Omega, |x-y| < 1, |p(x) - p(y)| \le \omega(|x-y|), \tag{1.3}$$

where $\limsup_{\tau\to 0^+} \omega(\tau)\ln(1/\tau) = C < +\infty.$

The problems linked to (1.1) are prevalent in a wide range of modern physics and engineering, such as electrorheological fluids, viscosity in fluids which depends on temperature, viscoelastic materials, filtration through porous structures, and applications in image processing (see [1-4]). We refer the readers to [5-7] for more applications and details on the topic.

When $f(u)=|u|^{p-2}u,$ Problem (1.1) has been extensively investigated in recent decades, and many authors have examined the existence, nonexistence, and decay of solutions (see [8-9]). When $h\equiv 0,$ Berrimi and Messaoudi studied Problem (1.1) and they found that, depending on the decaying rate of the kernel $g,$ the global existence of a solution happens and the decaying rate will either be exponential or polynomial [8]. After that, Messaoudi also considered Problem (1.1) and obtained the decay properties of the solution that includes exponential and polynomial decay rates [9].

Recently, in the scenario where $h(u_t)=-\Delta u_t,$ and $f(u)=|u|^{p(x)-2}u,$ Nhan et al. [1] studied (1.1), and they derived the estimate on the decay rate for a global solution within the stable set initially. More precisely, under the conditions of the relaxation g as follows:

(A1) *the relaxation* $g\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ *and fulfills*

$$g(0)>0,\ \ell=1-\int_0^\infty g(\kappa)d\kappa>0,\ g'(t)\le 0,\ \textit{for each } t\ge 0,$$

(A2) *there is* $k>0$ *and a differentiable function* $\xi(t)>0$ *such that*

$$g'(t)\le -\xi(t)g(t),\ \xi'(t)\le 0,\ |\xi'(t)/\xi(t)|\le k,\ \int_0^\infty \xi(t)dt=+\infty,\ \forall t>0,$$

the authors obtained the following theorem.

Theorem 1.1. (See [1], Theorem 2.9) Assume that (1.2) and (1.3) hold. Let g satisfy (A1), (A2). Furthermore, assume that $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ with $u_0\in\mathcal{W}_\delta\ (0<\delta\le\ell)$ and $E(0)<2^{(p^- -2)/2}E_{d_\delta},$ where $E_{d_\delta}=\left(\frac{1}{2}-\frac{1}{p^-}\right)d_\delta\Big/\left(\frac{1}{2}-\frac{1}{p^+}\right).$ Let $u(t)$ be a solution to (1.1). Then $u(t)$ decays exponentially.

Herein, we intend to extend and improve the decay finding of Nhan et al. in [1] (Theorem 7) to the broad decay rates of total energy, including polynomial, exponential, and logarithmic rates. More precisely, we improve and generalize the above theorem in two approaches:

- Firstly, we extend the initial energy to the case of sub-critical initial energy data $E(0)<d_\delta,$
- Secondly, we consider a more general and relatively large class of relaxation functions, that is,

(A3) *There exists a function* $G\in C^1\left((0,\infty);(0,\infty)\right)$ *which satisfies either*

(i) G *is linear, or*

(ii) $G\in C^2\left((0,r]\right)$ *is strictly increasing and strictly convex for some* $r\le g(0),$ *with* $G(0)=G'(0)=0$ *and*

$$g'(t)\le -\xi(t)G\left(g(t)\right),\forall t\ge 0,$$

where ξ *is positive, nonincreasing and differentiable.*

We shall show that the decay rates for the relaxation kernel g will determine the decay rates of the total energy when the initial energy data is below the mountain pass level (Theorem 3.6).

2. PRELIMINARIES

2.1. Modified potential wells

Throughout this paper, we define the functionals J_δ and I_δ (for $0<\delta\le\ell$) as in [1, 3]

$$J_\delta(u)=\frac{\delta}{2}\|\nabla u\|^2-\int_\Omega\frac{|u|^{p(x)}}{p(x)}dx,\quad\text{and}\quad I_\delta(u)=\delta\ \|\nabla u\|^2-\int_\Omega|u|^{p(x)}dx,$$

the Nehari manifold

$$\mathcal{N}_\delta = \left\{u \in H_0^1(\Omega)\setminus\{0\} : I_\delta(u) = \langle J'_\delta(u), u\rangle = 0\right\},$$

the potential well depth

$$d_\delta = \inf_{u\in\mathcal{N}_\delta} J_\delta(u), \tag{2.1}$$

and the modified stable set as in [1, 3]

$$\mathcal{W}_\delta = \left\{u \in H_0^1(\Omega) : J_\delta(u) < d_\delta, I_\delta(u) > 0\right\} \cup \{0\}.$$

2.2. Definition and preparing results

We start by defining the notion of weak solutions to Problem (1.1).

Definition 2.2. For each $0 < T \le \infty$, we call u a weak solution to (1.1) on $\Omega\times(0,T)$ when

$$u \in C([0,T); H_0^1(\Omega)),\ u_t \in C([0,T); L^2(\Omega)) \cap L^2([0,T); H_0^1(\Omega)),$$

and satisfies $u(x,0) = u_0(x) \in H_0^1(\Omega), u_t(x,0) = u_1(x) \in L^2(\Omega)$ and the equality

$$\int_\Omega \left[u_{tt}(t)\varphi + \nabla u(t)\cdot\nabla\varphi - \left(\int_0^t g(t-\kappa)\nabla u(\kappa)d\kappa\right)\cdot\nabla\varphi + \nabla u_t(t)\cdot\nabla\varphi - |u(t)|^{p(\cdot)-2} u(t)\varphi \right] dx = 0,$$

holds for a.e. $t \in (0,T)$ and any $\varphi \in H_0^1(\Omega)$.

Define the energy functional

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t g(\kappa)d\kappa\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\square\nabla u)(t) - \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)} dx,$$

where $(g\square\nabla u)(t) = \int_0^t g(t-\kappa)\|\nabla u(t) - \nabla u(\kappa)\|^2 d\kappa.$ By testing (1.1) by u_t, we have

$$\frac{d}{dt}E(t) = -\frac{1}{2}g(t)\|\nabla u(t)\|^2 + \frac{1}{2}(g'\square\nabla u)(t) - \|\nabla u_t(t)\|^2 \le 0, \tag{2.2}$$

which yields that $E(t)$ is non-increasing.

The local existence of solution to (1.1) can be stated as follows.

Theorem 2.3. (Local existence) (see [1]) Suppose there hold (1.2), (1.3) and (G, (i)). Then for given $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, there exists a unique local solution to Problem (1.1) with

$$u \in C([0,T_{\max}); H_0^1(\Omega)),\ u_t \in C([0,T_{\max}); L^2(\Omega)) \cap L^2([0,T_{\max}); H_0^1(\Omega)),$$

where $T_{\max} > 0$ denotes the maximal life time of $u(t)$.

We ended this section with the following proposition which is essential for proving our main results.

Proposition 2.4. Let (1.2) – (1.3) and (A1) hold and $0 < \delta \le \ell$. Suppose that $u(t)$ solves Problem (1.1) locally. Suppose there exists $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{W}_\delta$ and $E(t_0) < d_\delta$. Then $u(t) \in \mathcal{W}_\delta$ for any $t \in [t_0, T_{\max})$.

Proof. First, it follows from the definition of the energy functions $E(t)$ and J_δ that

$$E(t) \ge \frac{\delta}{2}\|\nabla u(t)\|^2 - \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)} dx = J_\delta(u(t)),$$

due to $1-\int_0^t g(\tau)d\tau \ge 1-\int_0^\infty g(\tau)d\tau = \ell.$ And then

$$J_\delta(u(t_0)) \le E(t_0) < d_\delta. \tag{2.3}$$

We shall prove that $u(t)\in \mathcal{W}_\delta,$ for all $t\in[t_0,T_{\max}).$ In fact, if not, there must be a $t_0 < t_1^* \le T_{\max}$ such that $I_\delta(u(t_1^*))=0$ and $u(t_1^*)\ne 0,$ which yields $u(t_1^*)\in \mathcal{N}_\delta.$ And then, the definition of d_δ tells us that $J_\delta\left(u(t_1^*)\right)\ge d_\delta,$ which is a contradiction to (2.3) due to $J_\delta(u(t_1^*))\le E(t_1^*)\le E(t_0).$ The proposition is proved.

3. GENERAL DECAY

3.1. Preliminary results

Herein we present several lemmas which will be necessary for the decay estimates.

Lemma 3.1. (See [10], Remark 2.8) There exist positive constants d_* and $t_1 = g^{-1}(r)$ such that for all $0\le t\le t_1,$ one has

$$g'(t)\le -d_* g(t).$$

Lemma 3.2. (See [10], Lemma 3.2) Suppose (A1) and (A3) are valid. Then

$$\int_\Omega \left(\int_0^t g(t-\kappa)(v(t)-v(\kappa))d\kappa\right)^2 dx \le C_\sigma (h_\sigma \square v)(t)$$

for all $v\in L^2_{loc}\left(0,\infty; L^2(\Omega)\right),$ $0<\sigma<1,$ where

$h_\sigma(t)=\sigma g(t)-g'(t),$ $\left(h_\sigma \square v\right)(t)=\int_0^t h_\sigma(t-\kappa)\|v(t)-v(\kappa)\|^2 d\kappa$ and the constant

$$C_\sigma = \int_0^\infty \frac{g^2(\kappa)}{h_\sigma(\kappa)}d\kappa \le \frac{1}{\sigma}\int_0^\infty g(\kappa)d\kappa < \infty.$$

Now, we define the following auxiliary functionals

$$\Phi_1(t)=\langle u_t(t),u(t)\rangle, \tag{3.1}$$

$$\Phi_2(t)=-\left\langle u_t(t),\int_0^t g(t-\kappa)\big(u(t)-u(\kappa)\big)d\kappa\right\rangle. \tag{3.2}$$

Then the next two lemmas hold.

Lemma 3.3. Let (A1), (A3) hold and $\left(u_0,u_1\right)\in H_0^1(\Omega)\times L^2(\Omega)$ with $u_0\in\mathcal{W}_\delta$ and $E(0)<d_\delta.$ Then, for any $0<\sigma<1$ and $\eta>0,$ we get

$$\frac{d\Phi_1}{dt}\le \|u_t(t)\|^2+\frac{1}{2\eta}\|\nabla u_t(t)\|^2-\left[\ell-\ell\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}}-\eta\right]\|\nabla u(t)\|^2+\frac{C_\sigma}{2\eta}\left(h_\sigma\square\nabla u\right)(t).$$

Proof. Equation (1.1) multiplied by $u(t)$ yields that

$$\begin{aligned}\frac{d\Phi_1}{dt} &= \|u_t(t)\|^2 - \left(1-\int_0^t g(\kappa)d\kappa\right)\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^{p(x)}dx \\ &\quad -\langle \nabla u_t(t), \nabla u(t)\rangle + \int_0^t g(t-\kappa)\langle \nabla u(t)-\nabla u(\kappa), \nabla u(t)\rangle d\kappa \\ &\le \|u_t(t)\|^2 - \ell\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^{p(x)}dx - \langle \nabla u_t(t), \nabla u(t)\rangle \\ &\quad + \int_0^t g(t-\kappa)\langle \nabla u(t)-\nabla u(\kappa), \nabla u(t)\rangle d\kappa,\end{aligned} \tag{3.3}$$

due to $1-\int_0^t g(\kappa)d\kappa \ge 1-\int_0^\infty g(\kappa)d\kappa = \ell.$ First, by the same argument as in [1, the estimate 39], we have

$$\int_\Omega |u(t)|^{p(x)}\, dx \le \delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}} \|\nabla u(t)\|^2 \le \ell\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}} \|\nabla u(t)\|^2. \tag{3.4}$$

Then, thanks to Cauchy-Schwarz and Young inequality, we obtain

$$-\langle \nabla u_t(t), \nabla u(t)\rangle \le \frac{\eta}{2}\|\nabla u(t)\|^2 + \frac{1}{2\eta}\|\nabla u_t(t)\|^2, \tag{3.5}$$

for any $\eta > 0.$ Using again Cauchy-Schwarz and Young inequality and Lemma 3.2, we have

$$\begin{aligned}\int_0^t g(t-\kappa)\langle \nabla u(t)-\nabla u(\kappa), \nabla u(t)\rangle d\kappa &\le \frac{\eta}{2}\|\nabla u(t)\|^2 + \frac{1}{2\eta}\int_\Omega\left(\int_0^t g(t-\kappa)(\nabla u(t)-\nabla u(\kappa)d\kappa\right)^2 dx \\ &\le \frac{\eta}{2}\|\nabla u(t)\|^2 + \frac{C_\sigma}{2\eta}\left(h_\sigma \Box \nabla u\right)(t).\end{aligned}$$

From (3.3) – (3.5), we complete the proof.

Lemma 3.4. For any $0<\beta_1<1,$ let (A1), (A3) hold and $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ with $u_0\in \mathcal{W}_\delta$ and $E(0)<d_\delta,$ one has

$$\frac{d\Phi_2}{dt} \le \left(\beta_1 - \int_0^t g(\kappa)d\kappa\right)\|u_t(t)\|^2 + \frac{\beta_1}{2}\|\nabla u_t(t)\|^2 + \frac{\beta_1}{2}C(\ell)\|\nabla u(t)\|^2 + \frac{\bar{C}}{2\beta_1}(1+C_\sigma)\left(h_\sigma \Box \nabla u\right)(t), \tag{3.6}$$

where $C(\ell) = C+2-\ell,\ \bar{C} = \max\{h_\infty S_2^2, 2\beta_1+3-\ell+2S_2^2\}$ and the constant

$$h_\infty := \int_0^\infty h_\sigma(\kappa)d\kappa \le \sigma\int_0^\infty g(\kappa)d\kappa + g(0) < \infty.$$

Proof. Thanks to the definition of $\Phi_2,$ we get that

$$\begin{aligned}\Phi_2'(t) &= -\left\langle u_{tt}(t), \int_0^t g(t-\kappa)\left(u(t)-u(\kappa)\right)d\kappa\right\rangle - \left\langle u_t(t), \int_0^t g'(t-\kappa)\left(u(t)-u(\kappa)\right)d\kappa\right\rangle \\ &\quad -\left(\int_0^t g(\kappa)d\kappa\right)\|u_t(t)\|^2 = J_1 + J_2 + J_3.\end{aligned} \tag{3.7}$$

Multiplying the first equation in (1.1) by $\int_0^t g(t-\kappa)\left(u(t)-u(\kappa)\right)d\kappa$ and using Green's formula, we find that

$$
\begin{aligned}
J_1 &= -\int_0^t g(t-\kappa)\langle \nabla u(t), \nabla u(t)-\nabla u(\kappa)\rangle d\kappa \\
&\quad -\left\langle \int_0^t g(t-\kappa)\nabla u(\kappa)d\kappa, \int_0^t g(t-\kappa)\big(\nabla u(t)-\nabla u(\kappa)\big)d\kappa \right\rangle \\
&\quad -\left\langle \nabla u_t(t), \int_0^t g(t-\kappa)\big(\nabla u(t)-\nabla u(\kappa)\big)d\kappa \right\rangle \\
&\quad -\left\langle |u(t)|^{p(x)-2}u(t), \int_0^t g(t-\kappa)\big(u(t)-u(\kappa)\big)d\kappa \right\rangle = \sum_{i=1}^{4} J_{1i}.
\end{aligned} \tag{3.8}
$$

We now shall estimate J_{1i} for $i=\overline{1,4}$.

For J_{11}. Using the well-known Cauchy-Schwarz and Young inequalities as well as Lemma 3.2, we first determine that

$$
\begin{aligned}
J_{11} &\le \frac{\beta_1}{2}\|\nabla u(t)\|^2 + \frac{1}{2\beta_1}\int_\Omega\left(\int_0^t g(t-\kappa)(\nabla u(t)-\nabla u(\kappa))d\kappa\right)^2 dx \\
&\le \frac{\beta_1}{2}\|\nabla u(t)\|^2 + \frac{C_\sigma}{2\beta_1}(h_\sigma \Box \nabla u)(t).
\end{aligned} \tag{3.9}
$$

For J_{12}. We first rewrite

$$
J_{12} = \left\| \int_0^t g(t-\kappa)\big(\nabla u(t)-\nabla u(\kappa)\big)d\kappa \right\|^2 - \left(\int_0^t g(\kappa)d\kappa\right)\left\langle \nabla u(t), \int_0^t g(t-\kappa)\big(\nabla u(t)-\nabla u(\kappa)\big)d\kappa \right\rangle.
$$

By applying the Cauchy-Schwarz and Young inequalities again, we have

$$
-\left\langle \nabla u(t), \int_0^t g(t-\kappa)\big(\nabla u(t)-\nabla u(\kappa)\big)d\kappa \right\rangle \le \frac{\beta_1}{2}\|\nabla u(t)\|^2 + \frac{C_\sigma}{2\beta_1}(h_\sigma \Box \nabla u)(t),
$$

hence, we obtain

$$
J_{12} \le \frac{\beta_1(1-\ell)}{2}\|\nabla u(t)\|^2 + \left(1+\frac{1-\ell}{2\beta_1}\right)C_\sigma\big(h_\sigma \Box \nabla u\big)(t). \tag{3.10}
$$

For J_{13}. Thanks to Lemma 3.2, the Cauchy-Schwarz and Young inequalities, one has

$$
J_{13} \le \frac{\beta_1}{2}\|\nabla u_t(t)\|^2 + \frac{C_\sigma}{2\beta_1}(h_\sigma \Box \nabla u)(t). \tag{3.11}
$$

For J_{14}. Since $2 < p^- \le p(x) \le p^+ \le \dfrac{2(n-1)}{n-2}$, we can use the Sobolev's embedding theorem and obtain

$$
\begin{aligned}
J_{14} &\le \frac{\beta_1}{2}\int_\Omega |u(t)|^{2(p(x)-1)}dx + \frac{1}{2\beta_1}\int_\Omega\left(\int_0^t g(t-\kappa)\big(u(t)-u(\kappa)\big)d\kappa\right)^2 dx \\
&\le \frac{\beta_1}{2}\max\left\{\|u(t)\|_{2(p(\cdot)-1)}^{2(p^- -1)}, \|u(t)\|_{2(p(\cdot)-1)}^{2(p^+ -1)}\right\} + \frac{C_\sigma}{2\beta_1}\int_0^t h_\sigma(t-\kappa)\|u(t)-u(\kappa)\|^2 d\kappa \\
&\le \frac{\beta_1}{2}\max\left\{S_{2(p(\cdot)-1)}^{2(p^- -1)}\|\nabla u(t)\|^{2(p^- -1)}, S_{2(p(\cdot)-1)}^{2(p^+ -1)}\|\nabla u(t)\|^{2(p^+ -1)}\right\} + \frac{C_\sigma S_2^2}{2\beta_1}\big(h_\sigma \Box \nabla u\big)(t) \\
&\le \frac{\beta_1}{2}C\|\nabla u(t)\|^2 + \frac{C_\sigma S_2^2}{2\beta_1}\big(h_\sigma \Box \nabla u\big)(t).
\end{aligned} \tag{3.12}
$$

Here C is a positive constant defined as in [1, Line 23, Page 15].

From (3.8) – (3.12), we derive that

$$J_1 \le \frac{\beta_1}{2}\left(C+2-\ell\right)\|\nabla u(t)\|^2 + \frac{\beta_1}{2}\|\nabla u_t(t)\|^2 + \frac{(S_2^2+2\beta_1+3-\ell)C_\sigma}{2\beta_1}\left(h_\sigma \square \nabla u\right)(t). \quad (3.13)$$

For J_2. We rewrite

$$\begin{aligned} J_2 &= \int_\Omega u_t(t)\int_0^t h_\sigma(t-\kappa)(u(t)-u(\kappa))d\kappa dx - \int_\Omega u_t(t)\int_0^t \sigma g(t-\kappa)(u(t)-u(\kappa))d\kappa dx \\ &= J_{21}+J_{22}. \end{aligned} \quad (3.14)$$

By using Lemma 3.2, the Cauchy-Schwarz, Young and Poincaré inequalities, we get

$$\begin{aligned} J_{21} &\le \frac{\beta_1}{2}\int_\Omega |u_t(t)|^2\, dx + \frac{1}{2\beta_1}\int_\Omega\left(\int_0^t \sqrt{h_\sigma(t-\kappa)}\sqrt{h_\sigma(t-\kappa)}\,|u(t)-u(\kappa)|\,d\kappa\right)^2 dx \\ &\le \frac{\beta_1}{2}\|u_t(t)\|^2 + \frac{\int_0^t h_\sigma(\kappa)d\kappa}{2\beta_1}(h_\sigma \square u)(t) \le \frac{\beta_1}{2}\|u_t(t)\|^2 + \frac{h_\infty S_2^2}{2\beta_1}(h_\sigma \square \nabla u)(t), \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} J_{22} &\le \frac{\beta_1}{2}\int_\Omega |u_t(t)|^2\, dx + \frac{\sigma^2}{2\beta_1}\int_\Omega\left(\int_0^t g(t-\kappa)\,|u(t)-u(\kappa)|\,d\kappa\right)^2 dx \\ &\le \frac{\beta_1}{2}\|u_t(t)\|^2 + \frac{\sigma^2 C_\sigma S_2^2}{2\beta_1}(h_\sigma \square \nabla u)(t), \end{aligned} \quad (3.16)$$

where $h_\infty = \int_0^\infty h_\sigma(\kappa)d\kappa < \infty,$ due to

$$\int_0^t h_\sigma(\kappa)d\kappa \le \sigma\int_0^t g(\kappa)d\kappa + g(0) - g(t) < \sigma\int_0^\infty g(\kappa)d\kappa + g(0),\ \forall t \ge 0.$$

Therefore

$$J_2 \le \beta_1\|u_t(t)\|^2 + \frac{h_\infty S_2^2}{2\beta_1}(h_\sigma \square \nabla u)(t) + \frac{\sigma^2 C_\sigma S_2^2}{2\beta_1}(h_\sigma \square \nabla u)(t). \quad (3.17)$$

Finally, from (3.7), (3.13) and (3.17), we obtain (3.6). This finishes the proof.

Next, we consider the Lyapunov function $L(t)$ defined by

$$L(t) = KE(t) + K_1\Phi_1(t) + K_2\Phi_2(t), \quad (3.18)$$

for K, K_1, K_2 enough large, where $\Phi_1(t)$ and $\Phi_2(t)$ are defined as in (3.1) and (3.2). It is straightforward that $E(t)$ and $L(t)$ are equivalent, i.e., there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t). \quad (3.19)$$

To estimate $L'(t),$ we have the following lemma.

Lemma 3.5. Given $t_1 > 0.$ Then $L(t)$ in (3.19) satisfies

$$L'(t) \le -\|u_t(t)\|^2 - 4(1-\ell)\|\nabla u(t)\|^2 + \frac{1}{4}(g \square \nabla u)(t),\ \ \forall t \ge t_1. \quad (3.20)$$

Proof. Using the fact that $g(0) > 0$ and $0 < g \in C(0,\infty),$ we have

$$\int_0^t g(\kappa)d\kappa \ge \int_0^{t_1} g(\kappa)d\kappa = g_1 > 0,\quad \forall t \ge t_1.$$

Therefore, combining the definition of $L(t),$ (2.2), and the fact that $g' = \sigma g - h,$ by choosing $\beta_1 = K_2^{-1},$ Lemmas 3.3 and 3.4 lead to

$$\begin{aligned}
L'(t) &= KE'(t) + K_1\Phi_1'(t) + K_2\Phi_2'(t) \\
&\le -\left[K_2\left(g_1-\beta_1\right)-K_1\right]\|u_t(t)\|^2 - \left(K - \frac{K_1}{2\eta} - \frac{K_2\beta_1}{2}\right)\|\nabla u_t(t)\|^2 \\
&\quad - \left[K_1\left(\ell - \ell\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}} - \eta\right) - \frac{K_2\beta_1}{2}C(\ell)\right]\|\nabla u(t)\|^2 \\
&\quad + \frac{K}{2}\left(g'\square\nabla u\right)(t) + \left[\frac{K_2\bar{C}}{2\beta_1} + \left(\frac{K_1}{2\eta} + \frac{\bar{C}K_2}{2\beta_1}\right)C_\sigma\right]\left(h_\sigma\square\nabla u\right)(t) \\
&\le -\left(K_2g_1 - 1 - K_1\right)\|u_t(t)\|^2 - \left(K - \frac{K_1}{2\eta} - \frac{1}{2}\right)\|\nabla u_t(t)\|^2 \\
&\quad - \left[K_1\left(\ell - \ell\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}} - \eta\right) - \frac{1}{2}C(\ell)\right]\|\nabla u(t)\|^2 + \frac{\sigma K}{2}\left(g\square\nabla u\right)(t) \\
&\quad - \left[\frac{K}{2} - \frac{\bar{C}K_2^2}{2} - \left(\frac{K_1}{2\eta} + \frac{\bar{C}K_2^2}{2}\right)C_\sigma\right]\left(h_\sigma\square\nabla u\right)(t).
\end{aligned} \tag{3.21}$$

Since $E(0) < d_\delta$, we can take sufficiently small $\eta > 0$ and sufficiently large $K_1 > 0$ so that

$$\ell - \ell\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}} - \eta > 0, \text{ and } K_1\left(\ell - \ell\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p^-}{2}} - \eta\right) - \frac{1}{2}C(\ell) > 4(1-\ell),$$

and then we select K_2 large enough such that

$$K_2g_1 - 1 - K_1 > 1.$$

Next, observe that since $\dfrac{g^2(\kappa)}{\sigma g(\kappa) - g'(\kappa)} \le g(\kappa)$, we infer from the Lebesgue dominated convergence theorem that

$$\sigma C_\sigma = \int_0^\infty \frac{g^2(\kappa)}{\sigma g(\kappa) - g'(\kappa)}d\kappa \to 0, \quad \text{as} \quad \sigma \to 0.$$

And hence, we can select a number $0 < \sigma_0 < 1$ such that if $\sigma < \sigma_0$ then

$$\sigma C_\sigma < \frac{1}{8}\left(\frac{K_1}{2\eta} + \frac{\bar{C}K_2^2}{2}\right)^{-1}.$$

After that, by choosing sufficiently large K and σ such that

$$K - \frac{K_1}{2\eta} - \frac{1}{2} > 0, \quad \frac{K}{4} - \frac{\bar{C}K_2^2}{2} > 0 \quad \text{and} \quad \sigma = \frac{1}{2K} < \sigma_0$$

to imply

$$\frac{K}{2} - \frac{\bar{C}K_2^2}{2} - \left(\frac{K_1}{2\eta} + \frac{\bar{C}K_2^2}{2}\right)C_\sigma > 0.$$

And finally, we obtain (3.20). The lemma is proved.

3.2. Decay results

The purpose of this section is to illustrate the decaying phenomenon of the energy functional of Problem (1.1) which is driven by the decaying rates of g.

Theorem 3.6. (General decay) Suppose that conditions (1.2) – (1.3) hold and g satisfies assumptions (A1), (A3). Let $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ with $u_0\in\mathcal{W}_\delta$, $(0<\delta\le\ell)$ and $E(0)<d_\delta$.

$$E(t)\le m_1 G_0^{-1}\left(\bar{m}_1\int_0^t \xi(s)\,ds\right),\ \text{for all}\ t\ge 2g^{-1}(r),$$

where $G_0(t):=\int_t^\tau \frac{1}{sG'(s)}ds$ is a strictly decreasing, convex function on $(0,\,r]$, with $\lim_{t\to 0}G_0(t)=+\infty.$

Remark 3.7. (a) Suppose (A3)(ii) holds. Then G can be extend to a function $\bar{G}$, which is a strictly increasing and strictly convex C^2-function on $(0,\infty)$. In particular, for $t>r$, we can define $\bar{G}$ by

$$\bar{G}(t):=\frac{G''(r)}{2}t^2+(G'(r)-G''(r)r)t+\left(G(r)+\frac{G''(r)}{2}r^2-G'(r)r\right). \tag{3.22}$$

(b) Concerning the relaxation function g satisfying assumptions (A1) and (A3), several examples are provided by Mustafa in [10].

Proof of Theorem 3.6. First, from (3.20) we deduce that

$$L'(t)\le -c_1E(t)+c_2(g\square\nabla u)(t),\quad \forall t\ge t_1. \tag{3.23}$$

Setting $t_1=g^{-1}(r)$ and referring to (2.2) together with Lemma 3.1, we get, for all $t\ge t_1$,

$$\int_0^{t_1} g(\kappa)\int_\Omega |\nabla u(t)-\nabla u(t-\kappa)|^2\,dxd\kappa\le -\frac{1}{d_*}\int_0^{t_1} g'(\kappa)\int_\Omega |\nabla u(t)-\nabla u(t-\kappa)|^2\,dxd\kappa\le -a_3E'(t), \tag{3.24}$$

Therefore

$$\begin{aligned} L'(t)&\le -c_1E(t)+c_2\left(g\square\nabla u\right)(t)\\ &\le -c_1E(t)-c_3E'(t)+c_2\int_{t_1}^t g(\kappa)\int_\Omega |\nabla u(t)-\nabla u(t-\kappa)|^2\,dxd\kappa,\quad \forall t\ge t_1, \end{aligned} \tag{3.25}$$

and hence

$$F'(t)\le -c_1E(t)+c_2\int_{t_1}^t g(\kappa)\int_\Omega |\nabla u(t)-\nabla u(t-\kappa)|^2\,dxd\kappa,\quad \forall t\ge t_1, \tag{3.26}$$

where $F(t)=L(t)+c_3E(t)$. It is straightforward to verify the equivalence between $F(t)$ and $E(t)$. Now we put

$$\mathcal{L}(t)=L(t)+\int_\Omega\int_0^t f(t-\kappa)\,|\nabla u(\kappa)|^2\,d\kappa dx,$$

in which $f(t)=\int_t^\infty g(\kappa)d\kappa$. Applying Lemma 3.2 and [6, Lemma 4.1], we see that $\mathcal{L}$ is nonnegative and satisfies

$$\mathcal{L}'(t)\le -\|u_t(t)\|^2-(1-\ell)\|\nabla u(t)\|^2-(1/4)(g\square\nabla u)(t)\le -C_*E(t),\ \forall t\ge t_1, \tag{3.27}$$

for some positive constant C_*. Therefore

$$C_*\int_{t_1}^{t} E(s)ds \le \mathcal{L}(t_1)-\mathcal{L}(t)\le \mathcal{L}(t_1),$$

which implies that

$$\int_0^\infty E(s)ds<\infty. \tag{3.28}$$

From (3.27), it follows that

$$r(t)=\int_0^t \|\nabla u(t)-\nabla u(s)\|^2\, ds<\infty, \tag{3.29}$$

which enables us to take $0<\vartheta<1$ such that $\vartheta r(t)\le 1, \forall t\ge t_1$. In view of the assumption on function G and the Jensen inequality we arrive at

$$\begin{aligned} G\left(\vartheta\int_0^t g(t-\kappa)\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa\right) &= G\left(\frac{1}{r(t)}\vartheta\int_0^t \vartheta r(t) g(t-\kappa)\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa\right)\\ &\le \frac{1}{r(t)}\int_0^t G\left[\vartheta r(t)g(t-\kappa)\right]\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa. \end{aligned} \tag{3.30}$$

Using the convexity of G together with $G(0)=0$ and $\vartheta r(t)\le 1,$ we deduce

$$G\left[\vartheta r(t)g(t-\kappa)\right]\le \vartheta r(t)G(g(t-\kappa)).$$

Hence, it follows from (3.29) that

$$\begin{aligned} &\bar{G}\left(\vartheta\int_0^t g(t-\kappa)\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa\right)=G\left(\vartheta\int_0^t g(t-\kappa)\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa\right)\\ &\le -\vartheta\int_0^t G\left(g(t-\kappa)\right)\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa \le -\vartheta\int_0^t \frac{g'(t-\kappa)}{\xi(t-\kappa)}\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa\\ &\le -\frac{\vartheta}{\xi(t)}\int_0^t g'(t-\kappa)\|\nabla u(t)-\nabla u(\kappa)\|^2\, d\kappa = -\frac{\vartheta}{\xi(t)}(g'\square\nabla u)(t)\le -\frac{\vartheta}{\xi(t)}E'(t). \end{aligned}$$

In turn, we obtain

$$(g\square\nabla u)(t)\le \vartheta^{-1}\bar{G}^{-1}\left(-\vartheta E'(t)/\xi(t)\right), \tag{3.31}$$

where $\bar{G}$ denotes an extension of G as in Remark 3.7(a). This in combination with (3.26) and (3.31) yields that

$$F'(t)\le -c_1E(t)+c_2(g\square\nabla u)(t)\le -c_1E(t)+c_2\vartheta^{-1}\bar{G}^{-1}\left(-\vartheta\frac{E'(t)}{\xi(t)}\right),\ \forall t\ge t_1. \tag{3.32}$$

For $0<\vartheta_1<r,$ it follows from (3.32) together with the facts that $E'\le 0, \bar{G}'>0$ and $\bar{G}''>0$ that

$$\mathcal{F}(t)=\bar{G}\left(\vartheta_1\frac{E(t)}{E(0)}\right)F(t)\sim E(t)>0.$$

Moreover, we get

$$\begin{aligned} \mathcal{F}'(t) &= \vartheta_1\frac{E'(t)}{E(0)}\bar{G}''\left(\vartheta_1\frac{E(t)}{E(0)}\right)F(t)+\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)F'(t)\\ &\le -c_1\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)E(t)+c_2\vartheta^{-1}\bar{G}\left(\vartheta_1\frac{E(t)}{E(0)}\right)\bar{G}^{-1}\left(-\vartheta\frac{E'(t)}{\xi(t)}\right),\quad \forall t\ge t_1. \end{aligned} \tag{3.33}$$

An application of Young's inequality yields

$$\begin{aligned}\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)\bar{G}^{-1}\left(-\vartheta\frac{E'(t)}{\xi(t)}\right) &\le \bar{G}^*\left(\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)\right)-\vartheta\frac{E'(t)}{\xi(t)} \\ &= \bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)(\bar{G}')^{-1}\left[\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)\right]-\bar{G}\left[(\bar{G}')^{-1}\left(\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)\right)\right]-\vartheta\frac{E'(t)}{\xi(t)},\end{aligned} \tag{3.34}$$

where

$$\bar{G}^*(s) := s(\bar{G}')^{-1}(s)-\bar{G}\left[(\bar{G}')^{-1}(s)\right]$$

satisfies Young's inequality

$$AB \le \bar{G}^*(A)+\bar{G}(B).$$

Here we used $A=\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)$ and $B=\bar{G}^{-1}\left(-\vartheta\frac{E'(t)}{\xi(t)}\right)$ in (3.34).

Next we use (3.34) and the facts that $\vartheta_1\frac{E(t)}{E(0)}<r$ and $\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)=G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)$ to derive

$$\begin{aligned}\bar{G}'\left(\vartheta_1\frac{E(t)}{E(0)}\right)\bar{G}^{-1}\left(-\vartheta\frac{E'(t)}{\xi(t)}\right) &\le \vartheta_1\frac{E(t)}{E(0)}G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)-G\left(\vartheta_1\frac{E(t)}{E(0)}\right)-\vartheta\frac{E'(t)}{\xi(t)} \\ &\le \vartheta_1\frac{E(t)}{E(0)}G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)-\vartheta\frac{E'(t)}{\xi(t)}.\end{aligned} \tag{3.35}$$

From (3.33) and (3.35), we infer that

$$\begin{aligned}\mathcal{F}'(t) &\le -c_1G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)E(t)+c_2\vartheta^{-1}G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)G^{-1}\left(-\vartheta\frac{E'(t)}{\xi(t)}\right) \\ &\le -c_1G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)E(t)+c_2\vartheta^{-1}\vartheta_1\frac{E(t)}{E(0)}G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)-c_2\frac{E'(t)}{\xi(t)},\quad \forall t\ge t_1,\end{aligned}$$

and thus

$$\xi(t)\mathcal{F}'(t)\le -\left(c_1E(0)-c_2\vartheta_1\vartheta^{-1}\right)\xi(t)\frac{E(t)}{E(0)}G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)-c_2E'(t). \tag{3.36}$$

Set $\mathcal{F}_1(t)=\xi(t)\mathcal{F}(t)+c_2E(t)$. Then $\mathcal{F}_1(t)\sim E(t)$ and for all $t\ge t_1$,

$$\begin{aligned}\mathcal{F}_1'(t) &= \xi(t)\mathcal{F}'(t)+\xi'(t)\mathcal{F}(t)+c_2E'(t)\le \xi(t)\mathcal{F}'(t)+c_2E'(t) \\ &\le -\left(c_1E(0)-c_2\vartheta_1\vartheta^{-1}\right)\xi(t)\frac{E(t)}{E(0)}G'\left(\vartheta_1\frac{E(t)}{E(0)}\right)\equiv -c_3\xi(t)\bar{G}_0\left(\vartheta_1\frac{E(t)}{E(0)}\right),\end{aligned} \tag{3.37}$$

where $\bar{G}_0(s):=sG'(s)$ is increasing. Then we get

$$\mathcal{F}_2'(t)\le -c_3\vartheta_2\xi(t)\bar{G}_0\left(\vartheta_1 E(t)/E(0)\right)\equiv -\bar{m}_1\xi(t)\bar{G}_0\left(\mathcal{F}_2(t)\right),\ \forall t\ge t_1, \tag{3.38}$$

where $\mathcal{F}_2(t)=\vartheta_2\left(\mathcal{F}_1(t)/E(0)\right)\sim E(t)$, for any $0<\vartheta_2<\vartheta_1$. Integrating (3.38) over $[t_1,t]$ to obtain

$$\int_{t_1}^{t}-\frac{\mathcal{F}_2'(s)}{\bar{G}_0\left(\mathcal{F}_2(s)\right)}ds\ge \bar{m}_1\int_{t_1}^{t}\xi(s)ds,\ \forall t\ge t_1,$$

which implies that

$$\int_{\mathcal{F}_2(t)}^{\mathcal{F}_2(t_1)}ds/\bar{G}_0(s)\ge \bar{m}_1\int_{t_1}^{t}\xi(s)ds,\ \ \forall t\ge t_1.$$

Therefore

$$\mathcal{F}_2(t) \le G_0^{-1}\left(\bar{m}_1 \int_{t_1}^{t} \xi(s)ds\right), \quad \forall t \ge t_1, \tag{3.39}$$

where $G_0(t) = \int_t^r ds/\bar{G}_0(s)$ defined on $\left(0, r\right]$ is strictly decreasing and $\lim_{t\to 0^+} G_0(t) = \infty.$ Here we make use of the properties of G and select ϑ_2 such that $\mathcal{F}_2(t_1) = \vartheta_2 \mathcal{F}_1(t_1) / E(0) \le r.$ In addition, by choosing $\bar{m}_1 < m_1$ satisfies $m_1 \int_{t_1}^{2t_1} \xi(s)ds = \bar{m}_1 \int_0^{2t_1} \xi(s)ds,$ we arrive at

$$m_1 \int_{t_1}^{t} \xi(s)ds \ge \bar{m}_1 \int_0^{t} \xi(s)ds. \tag{3.40}$$

From (3.39) and (3.40), we complete the proof.

4. CONCLUSION

Throughout the paper, under a relatively large class of relaxation g, we showed that the total energy of problem (1.1) satisfied the general decay rates, which include exponential, logarithmic, and polynomial rates.

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TÓM TẮT

TỐC ĐỘ TẮT DẦN TỔNG QUÁT CỦA NĂNG LƯỢNG TOÀN PHẦN CỦA MỘT PHƯƠNG TRÌNH SÓNG PHI TUYẾN CHỨA SỐ HẠNG TẮT DẦN MẠNH VỚI NGUỒN CHỨA SỐ MŨ BIẾN

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Bài báo này tập trung vào tốc độ tắt dần tổng quát của năng lượng toàn phần liên kết với nghiệm của phương trình sóng đàn hồi nhớt phi tuyến

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p(x)-2}u,$$

trên một miền bị chặn $\Omega \subset \mathbb{R}^n$, với giả thiết $g'(t) \le -\xi(t)G(g(t))$ trên hàm hồi phục với năng lượng ban đầu dưới ngưỡng tới hạn. Những kết quả này tinh chỉnh và mở rộng các kết quả gần đây của chúng tôi về phương trình sóng đàn hồi nhớt.

Từ khóa: Phương trình sóng phi tuyến, Đàn hồi nhớt, Tắt dần tổng quát, Tắt dần mạnh, Nguồn số mũ thay đổi.