REDUCTION METHODS PRESERVING STRUCTURED SYSTEMS - A PARALLEL INTERCONNECTION CASE

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ABSTRACT

In this paper, we investigate the model reduction problem for structured system by balanced truncation methods. These methods require the compromise between the reduction of each component and that of the whole structured system.

Keywords: Structured systems, parallel structure, model reduction, balanced truncation, H_∞-norm error bound.

1. INTRODUCTION

Structured systems have many applications, especially in physic, biology, chemistry, engineering, and so forth. The model reduction problems for structured systems are attracting research, see [1, 2] for example. The difficult point of the problem is that, for structured systems, one should care much about the interconnection of components inside the systems. Some reduction methods such as balanced truncation [3] may destroy the interconnection of structured systems. The use of reduction method should be careful when working with structured systems, especially with each component, with interconnection structure, and with the whole system itself. The aim of this paper is showing how to compromise between these factors when reducing structured systems. The reduction method we work with is balanced truncation, and the structured systems we choose is interconnected by parallel structure.

In general, the model reduction problems for structured systems can be stated as follows: let $G = F(G_1,...,G_k)$ be a structured system with k components $G_i, i = 1,...,k$ and the relation between these components is given by a function F . We want to find k reduced components $\hat{G}_i, i = 1,...,k$ such that with the same relation F the reduced structured system is formed as

$$\hat{G} = F(\hat{G}_1,...,\hat{G}_k)$$

satisfy that

$$\| G - \hat{G} \| \to \min$$

and

$$\| G_i - \hat{G}_i \| \to \min$$

for $i = 1,...,k$ and for some suitable norms of system, for example, H_∞-norm, or H_2-norm. We consider the structured systems we choose is interconnected by parallel structure, for the case of two components, see Figure 1. The case of many component is similar.

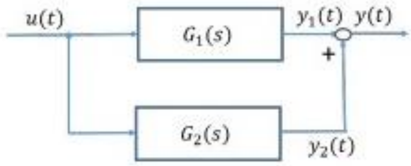

Figure 1. Structured system interconnected by parallel structure

In particular, let G_1 and G_2 be given by the following dynamical state-space equations

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t), \tag{1.1}$$

$$y_i(t) = C_i x_i(t) + D_i u(t), \tag{1.2}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m}, C_i \in \mathbb{R}^{p \times n_i}$ and $D_i \in \mathbb{R}^{p \times m}$. Here $i \in \{1,2\}$ indicate the indices of two systems G_1 and G_2, and $x_i(t) \in \mathbb{R}^m, i \in \{1,2\}$ are called the states of systems, $u(t) \in \mathbb{R}^m$ is input, and $y_i(t) \in \mathbb{R}^p, i \in \{1,2\}$ are outputs. The transfer function of two systems are matrix-valued complex functions defined as

$$G_i(s) := D_i + C_i(sI - A_i)^{-1} B_i, \text{where } s \in \mathbb{C}, i \in \{1,2\}.$$

Let $G(s) := G_1(s) + G_2(s)$ be a transfer function of structured system connected by G_1 and G_2 as Figure 1. The structured system G is given by the following dynamical state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.3}$$

$$y(t) = Cx(t) + Du(t), \tag{1.4}$$

where $x(t) \in \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^{n_1+n_2}$ is the state of G, $u(t) \in \mathbb{R}^m$ is input, and $y(t) = y_1(t) + y_2(t) \in \mathbb{R}^p$ is output. Coefficient matrices A, B, C, D are given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D = D_1 + D_2. \tag{1.5}$$

We introduce the H_∞-norm of system as follows. We assume that system G is asymptotically stable, that means matrix A is Hurwitz, i.e., $\lambda(A) \subset \mathbb{C}_-$. The H_∞-norm of G is defined by

$$\|G\|_{H_\infty} := \max_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega)), \tag{1.6}$$

where $\sigma_{\max}(G(j\omega))$ is the largest singular value of $G(j\omega)$. In the case that system G has single input, single output (SISO), the transfer function $G(j\omega)$ is just a complex number, therefore $\sigma_{\max}(G(j\omega)) = |G(j\omega)|$, which gives

$$\|G\|_{H_\infty} := \max_{\omega \in \mathbb{R}} |G(j\omega)|, \tag{1.7}$$

Now we state the model reduction problem that will be solved in this paper: Given two stable systems G_1 and G_2 described by (1.1)-(1.2), and G described by (1.3)-(1.5), find reduced order stable systems $\hat{G}_1$ and $\hat{G}_2$ such that the following three error systems have small H_∞-norm:

(a) $\|G_1 - \hat{G}_1\|_{H_\infty}$ is small,

(b) $\|G_2 - \hat{G}_2\|_{H_\infty}$ is small,

(c) $\|G - \hat{G}\|_{H_\infty}$ is small,

where $\hat{G} = \hat{G}_1 + \hat{G}_2$.

We will discuss two balanced truncation methods to solve the above problem. Each method is associated with a pair of Gramians, controllability and observability Gramians. For that the method that satisfies (a)-(b) is Algorithm 1, and the method that satisfies (c) is Algorithm 2-3. Besides, the error bound formula of these balanced truncation methods are also discussed.

The outline of this paper is as follows. In Section 2, we introduce algorithm for reducing the sum of two systems by balanced truncation method. In Section 3, other idea of balanced truncation method is considered, which firstly reduce the whole system and then construct the component. Numerical example and conclusion will be given in Section 4 and Section 5.

2. METHOD 1 - REDUCE EACH COMPONENT AND CONNECT THEM TOGETHER

2.1. Controllability and observability Gramians

Assume that G_1 and G_2 are two asymptotically stable systems described by (1.1)-(1.2). Then the following algebraic Lyapunov equations

$$A_i P_i + P_i A_i^T + B_i B_i^T = 0, \tag{2.1}$$

$$A_i^T Q_i + Q_i A_i + C_i^T C_i = 0, \; i \in \{1,2\}, \tag{2.2}$$

have unique positive definite solutions $(P_i, Q_i), i \in \{1,2\}$. Matrices P_i Q_i are called the controllability and observability Gramians of system G_i, $i \in \{1,2\}$.

2.2. Interconnected Systems Balanced Truncation Algorithm

Algorithm 1 is known as Interconnected Systems Balanced Truncation (ISBT) Algorithm appeared in [4].

Algorithm 1 Interconnected Systems Balanced Truncation [4]

Require: G_1 and G_2 are asymptotically stable two systems described by (1.1)-(1.2); and r is the reduced order.

Ensure: Two reduced subsystems $\hat{G}_1$ and $\hat{G}_2$, and the reduced system given by $\hat{G} = \hat{G}_1 + \hat{G}_2$.

1: Transforming (A_1, B_1, C_1, D_1) into balanced realization.
Let $(A_{1b}, B_{1b}, C_{1b}, D_{1b})$ be balanced realization and $\sigma_1^{(1)} \geq \sigma_2^{(1)} \geq \cdots \geq \sigma_{n_1}^{(1)} > 0$ be the Hankel singular values of system $G_1(s)$.

2: Transforming (A_2, B_2, C_2, D_2) into balanced realization.
Let $(A_{2b}, B_{2b}, C_{2b}, D_{2b})$ be balanced realization and $\sigma_1^{(2)} \geq \sigma_2^{(2)} \geq \cdots \geq \sigma_{n_2}^{(2)} > 0$ be the Hankel singular values of system $G_2(s)$.

3: Mix two sequences $\left\{\sigma_1^{(1)},\sigma_2^{(1)},\ldots,\sigma_{n_1}^{(1)}\right\}$, $\left\{\sigma_1^{(2)},\sigma_2^{(2)},\ldots,\sigma_{n_2}^{(2)}\right\}$ together and arrange the mixed sequence in decreasing order. Keep the first r states and truncate the last (n_1+n_2-r) states in the mixed sequence. Assume that the truncated states in the mixed sequence are corresponding to (n_1-r_1) last states in the first sequence and (n_2-r_2) last states in the second sequence, where $r=r_1+r_2$, as follows:

$$\underbrace{\sigma_1^{(1)}\ge\cdots\ge\sigma_{r_1}^{(1)}}_{r_1\text{ states}}>\underbrace{\sigma_{r_1+1}^{(1)}\ge\cdots\ge\sigma_{n_1}^{(1)}}_{(n_1-r_1)\text{ truncated states}},$$

$$\underbrace{\sigma_1^{(2)}\ge\cdots\ge\sigma_{r_2}^{(2)}}_{r_2\text{ states}}>\underbrace{\sigma_{r_2+1}^{(2)}\ge\cdots\ge\sigma_{n_2}^{(2)}}_{(n_2-r_2)\text{ truncated states}}.$$

4: Partition $(A_{1b},B_{1b},C_{1b},D_{1b})$ as

$$A_{1b}=\begin{bmatrix}\hat{A}_1 & *\\ * & *\end{bmatrix},\ B_{1b}=\begin{bmatrix}\hat{B}_1\\ *\end{bmatrix},$$

$$C_{1b}=\begin{bmatrix}\hat{C}_1 & *\end{bmatrix},\ D_{1b}=\hat{D}_1,$$

and $(A_{2b},B_{2b},C_{2b},D_{2b})$ as

$$A_{2b}=\begin{bmatrix}\hat{A}_2 & *\\ * & *\end{bmatrix},\ B_{2b}=\begin{bmatrix}\hat{B}_2\\ *\end{bmatrix},$$

$$C_{2b}=\begin{bmatrix}\hat{C}_2 & *\end{bmatrix},\ D_{2b}=\hat{D}_2,$$

where $\left(\hat{A}_1,\hat{B}_1,\hat{C}_1,\hat{D}_1\right)\in\mathbb{R}^{r_1\times r_1}\times\mathbb{R}^{r_1\times 1}\times\mathbb{R}^{1\times r_1}\times\mathbb{R}$ and $\left(\hat{A}_2,\hat{B}_2,\hat{C}_2,\hat{D}_2\right)\in\mathbb{R}^{r_2\times r_2}\times\mathbb{R}^{r_2\times 1}\times\mathbb{R}^{1\times r_2}\times\mathbb{R}$

5: Obtain two reduced systems of $G_1(s)$ and $G_2(s)$ respectively,

$$\hat{G}_1(s):=\hat{C}_1\left(sI-\hat{A}_1\right)^{-1}\hat{B}_1+\hat{D}_1, \tag{2.3}$$

$$\hat{G}_2(s):=\hat{C}_2\left(sI-\hat{A}_2\right)^{-1}\hat{B}_2+\hat{D}_2, \tag{2.4}$$

$$\hat{G}(s)\ :=\hat{C}\left(sI-\hat{A}\right)^{-1}\hat{B}+\hat{D}. \tag{2.5}$$

2.3. Error bounds

Theorem 1. *For reduced systems $\hat{G}_1,\hat{G}_2$, and $\hat{G}$ obtained from Algorithm 1, the following estimations hold:*

(a) $\left\|G_1-\hat{G}_1\right\|_{H_\infty}\le 2\left(\sigma_{r_1+1}^{(1)}+\cdots+\sigma_{n_1}^{(1)}\right)$,

(b) $\left\|G_2-\hat{G}_2\right\|_{H_\infty}\le 2\left(\sigma_{r_2+1}^{(2)}+\cdots+\sigma_{n_2}^{(2)}\right)$,

(c) $\left\|G-\hat{G}\right\|_{H_\infty}\le 2\left(\sigma_{r_1+1}^{(1)}+\cdots+\sigma_{n_1}^{(1)}\right)+2\left(\sigma_{r_2+1}^{(2)}+\cdots+\sigma_{n_2}^{(2)}\right)$

Proof. The parts (a) and (b) are the standard results of balanced truncation method [3]. For (c), it is obvious that

$$\left\|G(s)-\left(\hat{G}_1(s)+\hat{G}_2(s)\right)\right\|_{H_\infty} \le \left\|G(s)-\hat{G}_1(s)\right\|_{H_\infty} + \left\|G(s)-\hat{G}_2(s)\right\|_{H_\infty}$$
$$\le 2\left(\sigma^{(1)}_{r_1+1}+\cdots+\sigma^{(1)}_{n_1}\right)+2\left(\sigma^{(2)}_{r_2+1}+\cdots+\sigma^{(2)}_{n_2}\right).$$

3. METHOD 2 - REDUCE WHOLE SYSTEM AND RECONSTRUCT THE COMPONENT

3.1. Controllability and observability Gramians

Assume that G is asymptotically stable system described by (1.3)-(1.5). Then the following algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \tag{3.1}$$

$$A^TQ + QA + C^TC = 0, \tag{3.2}$$

have unique positive definite solutions (P,Q). Matrices P and Q are called the controllability and observability Gramians of system G.

3.2. Algorithms

Algorithm 2 Balanced Truncation Method [3]

Require: G_1 and G_2 are asymptotically stable two systems described by (1.1)-(1.2)); and r is the reduced order.

Ensure: The reduced system: $\hat{G}(s)$.

1: Transforming (A,B,C) with dimension (n_1+n_2).

$$(A,B,C)=\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_1 \end{bmatrix}\right),$$

into balanced realization. Let (A_b, B_b, C_b) be balanced realization and $\sigma_1 \ge \sigma_2 \ge \cdots \sigma_{n_1+n_2} > 0$ be the Hankel singular values of system $G(s)$.

2: Partition (A_b, B_b, C_b) as

$$A_b = \begin{bmatrix} \hat{A} & * \\ * & * \end{bmatrix}, B_b = \begin{bmatrix} \hat{B} \\ * \end{bmatrix}, C_b = \begin{bmatrix} \hat{C} & * \end{bmatrix},$$

where $(\hat{A},\hat{B},\hat{C}) \in \mathbb{R}^{r\times r} \times \mathbb{R}^{r\times 1} \times \mathbb{R}^{1\times r}$.

3: Obtain a reduced system

$$\hat{G}(s) = \hat{C}\left(sI-\hat{A}\right)^{-1}\hat{B}. \tag{3.3}$$

Next, from reduced system $\hat{G}(s)$ obtained in Algorithm 2 we propose the following algorithm to build two components $\hat{G}_1(s)$ and $\hat{G}_2(s)$ of $\hat{G}(s)$ as follows.

Algorithm 3 Reconstructing the component

Require: G_1 and G_2 are asymptotically stable two systems described by (1.1)-(1.2)); and r is the reduced order.

Ensure: The reduced systems $\hat{G}_1(s)$ and $\hat{G}_2(s)$ such that $\hat{G}(s)=\hat{G}_1(s)+\hat{G}_2(s)$.

1: Run Algorithm 2 to obtain $\hat{G}(s)$.

2: Compute the poles of $\hat{G}(s)$.

3: Compute the set of dominance poles of G_1 and G_2, say, Λ_1 and Λ_2.

4: Decide which poles of $\hat{G}(s)$ closed to Λ_1 or Λ_2

5: De-composite $\hat{G}(s)$ as the sum of two subsystem, $\hat{G}(s)=\hat{G}_1(s)+\hat{G}_2(s)$, where $\hat{G}_1(s)$ has poles closed to Λ_1 and $\hat{G}_2(s)$ has poles closed to Λ_2.

3.3. Error bounds

Theorem 2. *For reduced systems $\hat{G}_1,\hat{G}_2$, and $\hat{G}$ obtained from Algorithm 2-3, the following estimations hold:*

$$\left\|G-\hat{G}\right\|_{H_\infty} \le 2\left(\sigma_{r+1}+\cdots+\sigma_n\right).$$

Proof. See [3].

4. NUMERICAL EXAMPLE

4.1. Structured system

Consider a structured system G connected by G_1 and G_2 by parallel structure in a circuit simulation. The transfer functions $G_1(s)$ and $G_2(s)$ are given by

$$G_1(s)=\frac{s+4}{s^4+19s^3+113s^2+245s+150}$$

and

$$G_2(s)=\frac{2s^6+11.5s^5+57.75s^4+178.6s^3+345.5s^2+323.6s+94.5}{s^7+10s^6+46s^5+130s^4+239s^3+280s^2+194s+60}.$$

The set of poles of $G_1(s)$ and $G_2(s)$ are corresponding as follows $\Lambda_1=\{-10,-5,-3,-1\}$, and $\Lambda_2=\{-3,-2,-1,-1+i,-1-i,-1+2i,-1-2i,-1+3i,-1-3i\}$.

4.2. Algorithm 1

By taking $r=5$ and applying Algorithm 1 to G_1 and G_2, one obtains that from Matlab simulation

$$\hat{G}_1(s)=\frac{0.01396}{\text{s + 0.4378}},$$

$$\hat{G}_2(s) = \frac{1.961s^3 + 2.074s^2 + 27.74s + 13.24}{s^4 + 4.962s^3 + 12.41s^2 + 15.84s + 8.442},$$

and

$$\hat{G}(s) = \hat{G}_1(s) + \hat{G}_2(s) = \frac{1.975s^4 + 3.002s^3 + 28.83s^2 + 25.61s + 5.915}{s^5 + 5.4s^4 + 14.58s^3 + 21.27s^2 + 15.37s + 3.696}.$$

4.3. Algorithm 2

By taking $r = 5$ and applying Algorithm 2 to $G = G_1 + G_2$, one obtain that

$$\hat{G}(s) = \frac{2.001s^4 + 8.192s^3 + 35.52s^2 + 96.72s + 44.87}{s^5 + 8.358s^4 + 27.26s^3 + 54.9s^2 + 56.41s + 27.91}.$$

4.4. Algorithm 3

We compute the poles of $\hat{G}(s)$ obtained by Algorithm 2 as follows

$$\{-4.4228, -1.0575 + 1.8780i, -1.0575 - 1.8780i, -0.9099 + 0.7285i, -0.9099 - 0.7285i\}$$

Note that the set $\{-4.4228\}$ is closed to Λ_1, whereas the set

$$\{-1.0575 + 1.8780i, -1.0575 - 1.8780i, -0.9099 + 0.7285i, -0.9099 - 0.7285i\}$$

is closed to Λ_2. From that one could de-composite $\hat{G}(s)$ as $\hat{G}(s) = \hat{G}_1(s) + \hat{G}_2(s)$, where

$$\hat{G}_1(s) = \frac{1.9287}{s + 4.4228},$$

$$\hat{G}_2(s) = \frac{0.0719s^3 + 0.2845s^2 + 15.26s + 7.392}{s^4 + 3.935s^3 + 9.853s^2 + 11.33s + 6.311}.$$

4.5. H_∞ -norm of error systems

Numerically, the H_∞-norm of three error systems $G_1 - \hat{G}_1, G_2 - \hat{G}_2, G - \hat{G}$ are given in the following table. We note that Method 1 provides quite good result for the two first error systems, while Method 2 does for the last error system.

Table 1. The H_∞ -norm of three error systems $G_1 - \hat{G}_1, G_2 - \hat{G}_2, G - \hat{G}$

Error systems	Method 1 (Algorithm 1)	Method 2 (Algorithm 2, 3)
$\lVert G_1 - \hat{G}_1 \rVert_{H_\infty}$	0.0052	0.4133
$\lVert G_2 - \hat{G}_2 \rVert_{H_\infty}$	0.0118	0.4147
$\lVert G - \hat{G} \rVert_{H_\infty}$	0.0166	0.0057

5. CONCLUSION

The paper is dedicated to discuss two techniques that use balanced truncation methods for reducing the structured systems interconnected by parallel structure. These techniques provide small H_∞ -norm of error systems compromised between each component and the whole system itself.

In a forthcoming research, it is possible to extend to a wider class of structured systems, not only by parallel structure but also by series or combined structure.

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TÓM TẮT

PHƯƠNG PHÁP RÚT GỌN BẢO TOÀN HỆ CÓ CẤU TRÚC:
TRƯỜNG HỢP LIÊN KẾT SONG SONG

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Trong bài báo này tác giả sử dụng phương pháp chặt cân bằng để giải bài toán rút gọn mô hình cho hệ có cấu trúc. Những phương pháp này đòi hỏi có sự thỏa hiệp giữa việc rút gọn từng thành phần và việc rút gọn của toàn bộ hệ có cấu trúc.

Từ khóa: Hệ có cấu trúc, liên kết song song, rút gọn mô hình, phương pháp chặt cân bằng, sai số theo chuẩn H_∞ .